

A note on coherence power of N-dimensional unitary operators

M. García-Díaz, D. Egloff and M.B. Plenio

Institut für Theoretische Physik, Albert-Einstein-Allee 11, Universität Ulm, 89069 Ulm, Germany

(Dated: October 23, 2015)

The coherence power of a quantum channel, that is, its ability to increase the coherence of input states, is a fundamental concept within the framework of the resource theory of coherence. In this note we discuss various possible definitions of coherence power. Then we prove that the coherence power of a unitary operator acting on a qubit, computed with respect to the l_1 -coherence measure, can be calculated by maximizing its coherence gain over pure incoherent states. We proceed to show that this result fails for dimensions $N > 2$, that is, the maximal coherence gain is found when acting on a state with non-vanishing coherence.

I. INTRODUCTION

The development of quantum information science has led to a reassessment of quantum physical properties such as non-locality or entanglement, elevating them to *resources* that may be exploited to achieve tasks that are impossible when these properties are not available. The quantitative theory of entanglement [1, 2] was perhaps the first example of a theory that was formulated by taking seriously the idea that quantum properties are physical resources. The starting point was to take the view that constraints, here the restriction to local operations and classical communication, prevent certain non-local physical operations from being realizable unless resources, here entangled states, are available which may be consumed to allow us to overcome the imposed constraints [3, 4]. This viewpoint has proven fruitful as an impetus for theory to establish a unified and rigorously defined framework for a quantitative theory of physical resources by addressing the three principal issues: (i) the characterization, (ii) the quantification and (iii) the manipulation of quantum states under the imposed constraints. This framework is being explored for entanglement [1, 2], specific formulations of quantum thermodynamics [5, 6] and of reference frames [7, 8] and has led to the recognition of deep interrelations between the theories of entanglement and the second law [3, 4].

Recently, [9] formulated a resource theory for quantum coherence, which is a fundamental trait of quantum mechanics. In this work the authors defined a number of coherence measures and outlined, following the example of the theory of entanglement, various extensions that would have to be completed to explore all the aspects of the resource theory of coherence. This includes the study of the interconversion of coherent states by means of incoherent operations both, in the single copy [10–12] and the asymptotic regime [13] as well as the characterisation of incoherent operations [14, 15]. Although not addressed from the perspective of resource theory, [16, 17] have also dealt with the quantification of quantum coherence and the formal characterization of coherence-decreasing processes. The relationship between coherence and entanglement has been studied from various angles [18–20].

Aside of these developments it was pointed out in [9] that following the example of entanglement theory [21, 24] it would be natural to develop a quantitative theory of the coherence of operations which may have applications in the study

of coherence in dynamical processes including biological systems where the presence and role of coherence remains a matter of current debate [22, 23]. Indeed, first steps in this direction were taken in [25, 26] which mostly considered the coherence power of operations when acting on incoherent states. In our work we will demonstrate that while being consistent, this is too restrictive as it can be shown that the achievable coherence gain can be higher when accepting states as input which already possess some coherence [30]. This mirrors similar observations in the realm of entanglement theory [27, 28].

After this introduction, in section II of our manuscript we repeat some basic definitions concerning coherence measures which will be followed by a discussion of possible definitions of coherence properties of operations. This will be followed in section III by a discussion of the coherence power of unitaries on qubits which we prove to be achieved on incoherent states. Section IV then proceeds to demonstrate by means of two simple examples that for higher dimensional systems the largest gain in coherence is typically achieved on states with coherence. We conclude with a summary and outlook.

II. BASIC DEFINITIONS

In this section we provide the basic definitions of the quantities that we will be exploring in this work.

Measures of coherence of states – One result of the resource theory of coherence are well-defined quantifiers of coherence, coherence measures, which are quantities that cannot increase under the action of incoherent operations. Several such coherence measures could be identified and include the relative entropy of coherence as well as the l_1 -coherence [9]. While most definitions concerning the coherence power of operations can be formulated for any choice of coherence measure, for explicit calculations it is of advantage to consider the l_1 -coherence measure

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|. \quad (1)$$

Coherence properties of operations – Many physical questions relate to quantum operations and time evolution rather than directly to quantum states. Hence it is of considerable interest to examine the coherence properties of quantum operations or of their generators. Let us begin with the

Definition 1 The coherence power $P(\Phi)$ of a completely positive operation Φ is defined relative to the coherence measure $C(\cdot)$ via

$$P(\Phi) = \max_{\rho} [C(\Phi(\rho)) - C(\rho)]. \quad (2)$$

For a unitary operation the coherence power is therefore

$$P(U) = \max_{\rho} [C(U\rho U^\dagger) - C(\rho)]. \quad (3)$$

We have deliberately left unrestricted the range over which the ρ in the maximization are taken. In [25] this range was restricted to the set of incoherent states, i.e. the states for which $C(\rho) = 0$. While this may appear to be a natural choice it is not immediately clear that $C(\Phi(\rho)) - C(\rho)$ may actually be larger for some ρ with $C(\rho) > 0$. Indeed, motivated by similar observations in the theory of entanglement we consider this question and answer it in the affirmative [30] in section IV.

Of interest in the context of dynamical systems are the time dependent generalizations of the above concepts. Let us consider for example a time evolution $\Phi_t(\rho)$ with generator \mathcal{G} , that is $\Phi_t = e^{\mathcal{G}t}$ or for the special case of a unitary operator $U_t = e^{-iHt}$. Then one may either apply definition 1 at a time t or one may consider the coherence power of the generator by

Definition 2 For a time evolution $\Phi_t = e^{\mathcal{G}t}$ we determine the coherence power of the generator as

$$P(\mathcal{G}) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \max_{\rho} [C(e^{\mathcal{G}\Delta t}\rho) - C(\rho)] \quad (4)$$

and in case of unitary evolutions $U(t) = e^{-iHt}$ we write

$$P(H) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \max_{\rho} [C(e^{-iH\Delta t}\rho e^{iH\Delta t}) - C(\rho)]. \quad (5)$$

Note that one may also pursue questions concerning the coherence cost of an operation, that is, the amount of coherence in the form of maximally coherent states that is required to achieve an operation purely from incoherent operations. Questions regarding coherence cost and distillable coherence have been addressed in [13]. We will not pursue such quantities further here.

Of interest would be also to consider the N -dimensional unitary operations that have maximal coherence power. An example of this kind of unitaries would be the discrete Fourier transform:

Corollary 1 The coherence power of the discrete N -dimensional Fourier transform, calculated with respect to l_1 -coherence, is maximal and is given by:

$$P_{l_1}(\mathcal{F}) = N - 1 \quad (6)$$

Proof:

$$\begin{aligned} P_{l_1}(\mathcal{F}) &= \\ &= \max_{\rho} \left[\frac{1}{N} \sum_{a \neq b} \left| \sum_{j,j'} e^{\frac{2\pi i}{N}(ja-j'b)} \rho_{jj'} \right| - \sum_{a \neq b} |\rho_{ab}| \right] \\ &\geq \max_{\rho=|k\rangle\langle k|} \left[\frac{1}{N} \sum_{a \neq b} \left| \sum_{j,j'} e^{\frac{2\pi i}{N}(ja-j'b)} \rho_{jj'} \right| - \sum_{a \neq b} |\rho_{ab}| \right] \\ &= N - 1 \end{aligned}$$

Since $P_{l_1}(U) \leq N - 1$, we conclude that a discrete N -dimensional Fourier transform is an example of unitary having maximal coherence power.

III. COHERENCE POWER OF A 2-DIMENSIONAL UNITARY OPERATOR

As we have already mentioned, it is a non-trivial question whether it suffices in Definition 1 to restrict ρ to incoherent states or whether the full range of possible states, including states with coherence, need to be considered. First we formulate and prove

Theorem 1 The coherence power of a 2-dimensional unitary operation U acting on qubits and calculated with respect to the l_1 -coherence is maximal for pure incoherent states

$$P_{l_1}(U) = \max_{i=1,2} [C_{l_1}(U|i\rangle\langle i|U^\dagger)].$$

Proof: First we note that the coherence power of $R_z(\alpha)UR_z(\beta)$ is the same as that for U .

$$\begin{aligned} P(R_z(\alpha)UR_z(\beta)) &= \\ &= \max_{\rho} [C(R_z(\alpha)UR_z(\beta)\rho R_z^\dagger(\beta)U^\dagger R_z^\dagger(\alpha)) - C(\rho)] \\ &= \max_{\rho} [C(UR_z(\beta)\rho R_z^\dagger(\beta)U^\dagger) - C(\rho)] \\ &= \max_{\rho} [C(U\rho U^\dagger) - C(R_z^\dagger(\beta)\rho R_z(\beta))] \\ &= \max_{\rho} [C(U\rho U^\dagger) - C(\rho)] \\ &= P(U) \end{aligned}$$

Now consider

$$M = \begin{pmatrix} e^{i(\psi+\alpha)} & 0 \\ 0 & e^{-i(\psi-\alpha)} \end{pmatrix} \begin{pmatrix} u_{gg} & u_{ge} \\ u_{eg} & u_{ee} \end{pmatrix} \begin{pmatrix} e^{i(\phi+\beta)} & 0 \\ 0 & e^{-i(\phi-\beta)} \end{pmatrix}$$

where α and β are global phases without physical effect. We choose α and ψ such that $u_{gg}e^{i(\psi+\alpha)} \in \mathbb{R}^+$ and $u_{eg}e^{i(\psi-\alpha)} \in \mathbb{R}$. Hence we find

$$M = \begin{pmatrix} u_{gg} & u_{ge} \\ u_{eg} & u_{ee} \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{i\beta} \end{pmatrix} \quad (7)$$

with $u_{gg} \in \mathbb{R}^+$ and $u_{eg} \in \mathbb{R}$. Now choose $\phi = -\beta$ and make use of the orthonormality of the columns in a unitary

$$u_{gg}(u_{ge}e^{-2i\phi}) + u_{eg}(u_{ee}e^{-2i\phi}) = 0 \quad (8)$$

to conclude from $u_{gg}, u_{eg} \in \mathbb{R}$ that the phase of $u_{ge}e^{-2i\phi}$ and $u_{ee}e^{-2i\phi}$ is equal and can be eliminated by appropriate choice of ϕ . Hence we can assume

$$M = \begin{pmatrix} u_{gg} & u_{ge} \\ u_{eg} & u_{ee} \end{pmatrix} \quad (9)$$

with u_{gg}, u_{eg}, u_{ge} and $u_{ee} \in \mathbb{R}$. Hence we can start by considering real U and using $\rho_{gg} = 1 - \rho_{ee}$ and $\rho_{eg} = \rho_{ge}e^{i\gamma}$ we find

$$P(U) = 2 \max_{\rho} [|u_{ee}u_{ge} + \rho_{gg}(u_{eg}u_{gg} - u_{ee}u_{ge}) + \rho_{ge}(u_{ee}u_{gg} + e^{i\gamma}u_{eg}u_{ge})| - |\rho_{ge}|]$$

As the first two terms are real and the third term can be chosen to have any phase by virtue of the freedom of phase of ρ_{ge} we notice that the absolute value takes on its maximum value when $\rho_{ge}(u_{ee}u_{gg} + e^{i\gamma}u_{eg}u_{ge})$ is real and has the same sign as the sum of the first two terms.

Now let us choose $\rho_{ge}(u_{ee}u_{gg} + e^{i\gamma}u_{eg}u_{ge}) \in \mathbb{R}$ and with the same sign as $u_{ee}u_{ge} + \rho_{gg}(u_{eg}u_{gg} - u_{ee}u_{ge})$ (the case for opposite sign is treated analogously). Then there are two cases:

1) $u_{ee}u_{ge} + \rho_{gg}(u_{eg}u_{gg} - u_{ee}u_{ge}) > 0$ which leads to

$$P(U) = 2 \max_{\rho} [(u_{ee}u_{ge} + \rho_{gg}(u_{eg}u_{gg} - u_{ee}u_{ge}) + |\rho_{ge}|(|u_{ee}u_{ge} + e^{i\gamma}u_{eg}u_{ge}| - 1))]$$

As $U \in \mathbb{R}$ we have

$$|u_{ee}u_{gg} + e^{i\gamma}u_{eg}u_{ge}| = \left| \begin{pmatrix} u_{gg} \\ u_{ge}e^{i\gamma} \end{pmatrix} \begin{pmatrix} u_{ee} \\ u_{eg} \end{pmatrix}^\dagger \right|$$

As the vectors on the right are normalized the modulus of their scalar product is bounded by 1. Therefore $2|\rho_{ge}|(|u_{ee}u_{gg} + e^{i\gamma}u_{eg}u_{ge}| - 1) \leq 0$ and takes its maximum for $\rho_{ge} = 0$.

2) $u_{ee}u_{ge} + \rho_{gg}(u_{eg}u_{gg} - u_{ee}u_{ge}) < 0$ proceeds along the same lines.

The coherence power of a 2-dimensional unitary is therefore achieved for states ρ that are incoherent. To complete the proof of the theorem we now note that by the convexity $C(\sum_n p_n \rho_n) \leq \sum_n p_n C(\rho_n)$ for any set of states $\{\rho_n\}$ and probability distribution $\{p_n\}$ we find

$$\begin{aligned} C_{l_1}(U \rho_{inc} U^\dagger) &= C_{l_1}(U \sum_i p_i |i\rangle\langle i| U^\dagger) \\ &= C_{l_1}(\sum_i p_i U |i\rangle\langle i| U^\dagger) \\ &\leq \sum_i p_i C_{l_1}(U |i\rangle\langle i| U^\dagger) \\ &\leq C_{l_1}(U |i^*\rangle\langle i^*| U^\dagger) \end{aligned}$$

where $|i^*\rangle\langle i^*|$ is the pure incoherent state which has the largest contribution in the sum [29]. This concludes the proof. ■

From theorem 1 we easily find

Corollary 2 *The coherence power of a 2-dimensional unitary operation U , calculated with respect to the l_1 -coherence, is given by*

$$P_{l_1}(U) = \max_j \left\{ \left(\sum_{i=1}^2 |U_{ij}| \right)^2 : j = 1, 2 \right\} - 1 \quad (10)$$

Proof: Since in order to compute the coherence power of a 2-dimensional unitary we need to maximize the gain over pure incoherent states only, we find

$$\begin{aligned} P_{l_1}(U) &= \max_{|k\rangle\langle k|} [C_{l_1}(U |k\rangle\langle k| U^\dagger)] : k = 1, 2 \\ &= \max_j \left\{ \left(\sum_{i=1}^2 |U_{ij}| \right)^2 : j = 1, 2 \right\} - 1 \end{aligned}$$

■

IV. COHERENCE POWER OF AN N-DIMENSIONAL UNITARY OPERATOR ($N > 2$)

Naively it might be expected that the coherence power of any quantum channel is achieved on incoherent states. Indeed, the coherence power has been defined in this way in [25]. However, it is not self-evident that the largest coherence gain is obtained from incoherent states. Indeed, in the theory of entanglement the analogous question, i.e. whether the entanglement gain is maximized by starting on separable states, has been answered in the negative [27, 28]. In the following we show that the same observation holds for the case of coherence power.

Proposition 1 *For $N > 2$, the coherence power of an N -dimensional unitary operator requires optimization over coherent states.*

Proof: We consider the coherence power as quantified relative to the l_1 -coherence and the relative entropy of coherence [9].

l_1 -coherence power – Let us consider a 3-dimensional rotation by $\theta = \frac{\pi}{4}$ around the x axis:

$$R_x \left(\frac{\pi}{4} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (11)$$

According to Corollary 1, its maximum coherence gain calculated over pure incoherent states is found to be:

$$\max_j \left\{ \left(\sum_{i=1}^3 |R_x \left(\frac{\pi}{4} \right)_{ij}| \right)^2 : j = 1, 2, 3 \right\} - 1 = 1. \quad (12)$$

It is easy to find examples of coherent states that provide a larger coherence gain for this particular rotation. The state $|\psi\rangle = c_1|1\rangle + c_3|3\rangle$ where $c_1 = 0.3$ and $c_3 = \sqrt{1 - 0.3^2}$, for

instance, provides a coherence gain of 1.1471:

$$\begin{aligned}
G_{|\psi\rangle\langle\psi|}\left(R_x\left(\frac{\pi}{4}\right)\right) &= \\
&= C_{l_1}\left(\begin{array}{ccc} c_1^2 & \frac{c_1 c_3}{\sqrt{2}} & \frac{c_1 c_3}{\sqrt{2}} \\ \frac{c_1 c_3}{\sqrt{2}} & \frac{c_3^2}{2} & \frac{c_3^2}{2} \\ \frac{c_1 c_3}{\sqrt{2}} & \frac{c_3^2}{2} & \frac{c_3^2}{2} \end{array}\right) - C_{l_1}\left(\begin{array}{ccc} c_1^2 & 0 & c_1 c_3 \\ 0 & 0 & 0 \\ c_1 c_3 & 0 & c_3^2 \end{array}\right) \\
&= (2\sqrt{2} - 2)c_1 c_3 + c_3^2 \\
&= 1.1471 > 1.
\end{aligned}$$

Relative entropy of coherence power – Assuming that the coherence power could be calculated by maximization of the gain over incoherent states, and the observation that by convexity of the relative entropy of coherence we can then restrict maximization to pure incoherent states, we find for the coherence power of an N -dimensional unitary with respect to the relative entropy of coherence:

$$P_{rel.ent.}(U) = \max_i \left\{ -\sum_{j=1}^N |U_{ij}|^2 \log(|U_{ij}|^2) : i = 1, \dots, N \right\} \quad (13)$$

Proof:

$$\begin{aligned}
P_{rel.ent.}(U) &= \\
&= \max_{|i\rangle\langle i|} [C_{rel.ent.}(U|i\rangle\langle i|U^\dagger) - C_{rel.ent.}(|i\rangle\langle i|) : i = 1, \dots, N] \\
&= \max_{|i\rangle\langle i|} [S((U|i\rangle\langle i|U^\dagger)_{\text{diag}}) - S(U|i\rangle\langle i|U^\dagger) : i = 1, \dots, N] \\
&= \max_{|i\rangle\langle i|} [S((U|i\rangle\langle i|U^\dagger)_{\text{diag}}) : i = 1, \dots, N] \\
&= \max_i \left[-\sum_{j=1}^N |U_{ij}|^2 \log(|U_{ij}|^2) : i = 1, \dots, N \right].
\end{aligned}$$

Let us now consider a 3-dimensional rotation of $\theta = \frac{\pi}{8}$ around the x axis:

$$R_x\left(\frac{\pi}{8}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{\pi}{8}\right) & -\sin\left(\frac{\pi}{8}\right) \\ 0 & \sin\left(\frac{\pi}{8}\right) & \cos\left(\frac{\pi}{8}\right) \end{pmatrix} \quad (14)$$

Maximization of the coherence gain of this rotation over incoherent states results in

$$\begin{aligned}
\max_i \left\{ -\sum_{j=1}^3 \left| R_x\left(\frac{\pi}{8}\right)_{ij} \right|^2 \log\left(\left| R_x\left(\frac{\pi}{8}\right)_{ij} \right|^2\right) : i = 1, 2, 3 \right\} \\
= 0.41650.
\end{aligned}$$

However we have found a number of coherent states that provide an even larger gain, such as the state $|\phi\rangle = q_2|2\rangle + q_3|3\rangle$ where $q_2 = \sqrt{1 - 0.12533^2}$ and $q_3 = 0.12533$:

$$G_{|\phi\rangle\langle\phi|}\left(R_x\left(\frac{\pi}{8}\right)\right) = 0.47648 > 0.41650. \quad (15)$$

The maximum gain of these two rotations, with respect to their corresponding coherence measure, is not achieved on pure incoherent states. Therefore the most natural definition of the coherence coherence power is by maximization over all states.

V. CONCLUSION

In this note we have discussed several possible definitions of coherence power. We have also proved that the coherence power of a 2-dimensional unitary operator can be calculated by maximizing its coherence gain over pure incoherent states only. Giving two explicit counterexamples, we could show that this result cannot be generalized for dimensions higher than $N = 2$ [30].

Hence, analogously to the result of entanglement theory, where it was observed that entangled states typically admit the largest gain in entanglement, we found that some initial coherence in the input state can be required for an optimal coherence gain to be attained. This result shows that it is not sufficient to maximize the coherence gain over incoherent states. It seems therefore an interesting question if one can restrict the optimization in higher dimension to a smaller subset or one needs to run it over the whole state space even for unitary evolutions. For non-unitary evolutions, while it seems challenging to try to find a generic simplification, one still might use the symmetries present in coherence theory to simplify the optimization for a given evolution, similarly as we used them here in the case of qubits and unitary evolution for proving theorem 1.

ACKNOWLEDGMENTS

We acknowledge discussions with S.F. Huelga, N. Killoan, A. Smirne, M. Matera and K. Macieszczak. This work was supported by an Alexander von Humboldt Professorship, the EU Integrating Project SIQS as well as the EU STREPs EQUAM and QUCHIP.

[1] M.B. Plenio and S. Virmani, *Quant. Inf. Comput.* **7**, 1 (2007).
[2] V. Vedral and M.B. Plenio, *Phys. Rev. A* **57**, 1619 (1998).
[3] F.G.S.L. Brandão and M.B. Plenio, *Nature Phys.* **4**, 873 (2008).
[4] F.G.S.L. Brandão and M.B. Plenio, *Comm. Math. Phys.* **295**, 829 (2010).
[5] F.G.S.L. Brandão, M. Horodecki, J. Oppenheim, J.M. Renes and R.W. Spekkens, *Phys. Rev. Lett.* **111**, 250404 (2013).

[6] G. Gour, M.P. Müller, V. Narasimhachar, R.W. Spekkens, N. Younger Halpern, *Physics Reports* **583**, 1, (2015).
[7] G. Gour and R.W. Spekkens, *New J. Phys.* **10**, 033023 (2008); I. Marvian and R.W. Spekkens, *New J. Phys.* **15**, 033001 (2013).
[8] G. Gour, I. Marvian, and R.W. Spekkens, *Phys. Rev. A* **80**, 012307 (2009).

- [9] T. Baumgratz, M. Cramer, and M.B. Plenio, *Phys. Rev. Lett.* **113**, 140401 (2014).
- [10] S. Du, Z. Bai and Y. Guo, *Phys. Rev. A* **91**, 052120 (2015).
- [11] S. Du, Z. Bai and X. Qi, [E-print arXiv:1504.02862v2](#).
- [12] X. Qi, Z. Bai, and S. Du, [E-print arXiv:1505.07387v1](#).
- [13] A. Winter and D. Yang, [E-print arXiv:1506.07975v2](#).
- [14] A. Streltsov, S. Rana, M.N. Bera and M. Lewenstein, [E-print arXiv:1509.07456v2](#).
- [15] E. Chitambar and M.-H. Hsieh, [E-print arXiv:1509.07458v1](#).
- [16] J. Åberg, [E-print arXiv:quant-ph/0612146](#).
- [17] F. Levi and F. Mintert, *New J. Phys.* **16**, 033007 (2014).
- [18] N. Killoran, F.E.S. Steinhoff, and M.B. Plenio, [E-print arXiv:1505.07393v1](#).
- [19] A. Streltsov, U. Singh, H.S. Dhar, M.N. Bera and G. Adesso, *Phys. Rev. Lett.* **115**, 020403 (2015).
- [20] Y. Yao, X. Xiao, L. Ge and C.P. Sun, *Phys. Rev. A* **92**, 022112 (2015).
- [21] J. Eisert, K. Jacobs, P. Papadopoulos, and M.B. Plenio, *Phys. Rev. A* **62**, 052317 (2000).
- [22] S.F. Huelga and M.B. Plenio, *Contemp. Phys.* **54**, 181 (2013).
- [23] F. Caruso, A.W. Chin, A. Datta, S.F. Huelga, and M.B. Plenio, *Phys. Rev. A* **81**, 062346 (2010).
- [24] P. Zanardi, C. Zalka, and L. Faoro, *Phys. Rev. A* **62**, 030301 (2000).
- [25] A. Mani, V. Karimipour, *Phys. Rev. A* **92**, 032331 (2015).
- [26] K. Bu, L. Zhang, J. Wu, [E-print arXiv:1509.09109v1](#).
- [27] C.H. Bennett, A. Harrow, D.W. Leung, and J.A. Smolin, *IEEE Trans. Inf. Theo.* **49**, 1895 - 1911 (2003).
- [28] M. Leifer, L. Henderson and N. Linden, *Phys. Rev. A* **67**, 012306 (2003).
- [29] Note that this has also been observed in [25], where the coherence power of an arbitrary quantum channel calculated for any coherence measure is proved to be maximum for pure states, provided that the maximization is performed over incoherent states only.
- [30] After completion of this work we became aware of [26] which independently found that the coherence power is generally achieved only by maximization over the full state space and also has a different proof for theorem 1 and corollary 2.